

Differential equations

Any equation which contains $\frac{d^n y}{dx^n}$ in some form, for example, $\frac{dy}{dx} = 2x + 3y$, is called a differential equation. Here we are going to consider some of the many techniques used to *solve* these equations. Solving differential equations means finding $y = f(x)$.

Solution by direct integration

If we integrate $\frac{dy}{dx}$, then we get y and if we integrate $\frac{d^2 y}{dx^2}$, then we get $\frac{dy}{dx}$, etc. This enables us to solve differential equations of the form $\frac{d^n y}{dx^n} = f(x)$.

Example

Solve the differential equation $\frac{d^2 r}{dt^2} = 2t^2$.

Solution

Integrating once with respect to t :

$$\frac{dr}{dt} = \frac{2}{3}t^3 + c$$

Integrating again:

$$r = \frac{1}{6}t^4 + ct + d$$

As seen in this example, we can end up with several constants in the solution. This is called a 'general solution'. You can find a 'particular solution' if there are 'boundary conditions' given in the question, giving you particular values of the variables involved.

Example

Solve the differential equation $\frac{dy}{dx} = e^x$, given that $y = 1$ when $x = \ln 2$.

Solution

The general solution of the differential equation is $y = e^x + c$, but we know that when $x = \ln 2$, $y = 1$.

So:

$$1 = 2 + c \Rightarrow c = -1$$

and the particular solution is:

$$y = e^x - 1$$

Question 1.1

Find the general solution of the differential equation $\frac{d^2x}{dt^2} = at^2$. Given that when $t = 0$, $x = 4$, when $t = 1$, $x = \frac{97}{12}$, and when $t = -1$, $x = \frac{1}{12}$, find a and the particular solution.

Solution by separation of variables

If $\frac{dy}{dx} = f(x, y)$, and $f(x, y)$ can be expressed as $p(x)q(y)$, then the differential equation can be solved by separating the variables, ie by writing it in the form:

$$\int \frac{1}{q(y)} dy = \int p(x) dx$$

Example

Solve the differential equation $\frac{dr}{dt} = \frac{r}{2t+1}$, where $r > 0$ and $t > 0$.

Solution

We can rearrange this in the form $\frac{dr}{r} = \frac{dt}{2t+1}$, so:

$$\int \frac{1}{r} dr = \int \frac{1}{2t+1} dt$$

$$\ln r = \frac{1}{2} \ln(2t+1) + c \Rightarrow \ln r = \ln(\sqrt{2t+1}) + c$$

Note that we do not need to include modulus signs in the integrated expressions as both r and $2t+1$ are positive. Note also that we need to include a constant on one side of the equation, not both.

Taking exponentials of both sides:

$$e^{\ln r} = e^{\ln(\sqrt{2t+1}) + c} \Rightarrow r = e^c \sqrt{2t+1} \Rightarrow r = k\sqrt{2t+1}$$

where $k = e^c$.

Question 1.2

Find the particular solution of the differential equation $(x^2 + 9)\frac{dy}{dx} = xy + 2x$, where $y > 0$, given that when $x = 4, y = 0.5$.

Question 1.3

Find the general solution to the differential equation $\frac{dN}{dt} = (a - bN)N$, where $0 < N < \frac{a}{b}$.

This is known as the logistic equation or the Verhulst equation after the mathematician who used it in his study of populations.

Solution by integrating factor

A differential equation of the form $\frac{dy}{dx} + f(x)y = g(x)$ can be solved by use of an integrating factor. The integrating factor here is $\exp\left[\int f(x)dx\right]$, and the solution of the differential equation is then the solution of:

$$y \exp\left[\int f(x)dx\right] = \int g(x) \exp\left[\int f(x)dx\right] dx$$

Example

Find the general solution of $\frac{dy}{dx} + 4y = 2e^{3x}$.

Solution

The integrating factor is $\exp\left[\int 4 dx\right] = e^{4x}$. So we multiply through by e^{4x} which gives $e^{4x} \frac{dy}{dx} + 4ye^{4x} = 2e^{7x}$.

We see that the LHS is the derivative of ye^{4x} , derived from the product rule. So $\frac{d}{dx}(ye^{4x}) = 2e^{7x}$.

We then need to solve $ye^{4x} = \int 2e^{7x} dx$, which gives $ye^{4x} = \frac{2}{7}e^{7x} + c$. The solution of our differential equation is $y = \frac{2}{7}e^{3x} + ce^{-4x}$, where c is an arbitrary constant.

Question 1.4

Find the particular solution of the differential equation $3xy \frac{dy}{dx} = y^2 + 2xy$, where $x > 0$, given that $y = 1$ when $x = 27$.

Solutions

Solution 1.1

Integrating twice gives:

$$\frac{dx}{dt} = \frac{at^3}{3} + c$$

$$x = \frac{at^4}{12} + ct + d$$

But using the boundary conditions:

$$t = 0 \Rightarrow d = 4$$

$$t = 1 \Rightarrow \frac{97}{12} = \frac{a}{12} + c + 4$$

$$t = -1 \Rightarrow \frac{1}{12} = \frac{a}{12} - c + 4$$

Solving these simultaneously, we get $a = 1$, $c = 4$, so the particular solution is:

$$x = \frac{t^4}{12} + 4t + 4$$

Solution 1.2

Separating the variables gives us:

$$\int \frac{1}{y+2} dy = \int \frac{x}{x^2+9} dx$$

Integrating both sides gives us:

$$\ln(y+2) = c + \frac{1}{2} \ln(x^2+9)$$

Note that no modulus signs are required here as x^2+9 is always positive, and $y+2$ is always positive since we are given that $y > 0$.

So:

$$e^{\ln(y+2)} = e^{c+\ln\sqrt{x^2+9}} \Rightarrow y+2 = e^c \sqrt{x^2+9} \Rightarrow y = k\sqrt{x^2+9} - 2$$

where $k = e^c$. But if $y = 0.5$ when $x = 4$, then $k = 0.5$, so the particular solution is:

$$y = 0.5\sqrt{x^2+9} - 2$$

Solution 1.3

The differential equation can be solved by separating the variables:

$$\begin{aligned} \int \frac{dN}{(a-bN)N} &= \int dt \\ \int \frac{1/a}{N} + \frac{b/a}{a-bN} dN &= \int dt \\ \frac{1}{a} \ln N - \frac{1}{a} \ln(a-bN) + c &= t \end{aligned}$$

Note that $N > 0$ and $a - bN > 0$ from the condition in the question, so no modulus signs are required here.

Setting $c = \frac{1}{a} \ln k$, for some positive constant k , this can be simplified to give the general solution:

$$\frac{1}{a} \ln \frac{kN}{a-bN} = t \Rightarrow \frac{kN}{a-bN} = e^{at} \Rightarrow N = \frac{ae^{at}}{k + be^{at}}$$

Solution 1.4

The differential equation can be rewritten in the form $\frac{dy}{dx} - \frac{1}{3x}y = \frac{2}{3}$, providing that $y \neq 0$. $y=0$ is a trivial solution to the differential equation.

Using this version, we can see that the integrating factor is

$$\exp\left[\int -\frac{1}{3x} dx\right] = \exp\left[-\frac{1}{3} \ln x\right] = x^{-\frac{1}{3}}.$$

Note that since $x > 0$, we don't need to include modulus signs here.

Multiplying through by the integrating factor and integrating both sides with respect to x gives the general solution to the differential equation:

$$yx^{-\frac{1}{3}} = \int \frac{2}{3} x^{-\frac{1}{3}} dx = x^{\frac{2}{3}} + c$$

But we know that $y=1$ when $x=27$, so:

$$\frac{1}{3} = 9 + c \Rightarrow c = -8\frac{2}{3}$$

So the particular solution is $yx^{-\frac{1}{3}} = x^{\frac{2}{3}} - 8\frac{2}{3}$ or $y = x - 8\frac{2}{3}x^{\frac{1}{3}}$.